Department of Mathematics & Philosophy of Engineering

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Course: MPZ 3231-Engineering Mathematics IA **Model Answer No·03**

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1) Let $z = \cos\theta + i\sin\theta \quad \theta \in \mathbb{R}$,

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad n \in \mathbb{Z}^+$

Let $n < 0 \qquad n = -m \qquad m > 0$

$$z^n = (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$
$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta} \qquad m \in \mathbb{Z}^+$$
$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$
$$= \cos m\theta - i\sin m\theta$$
$$= \cos(-m\theta) + i\sin(-m\theta)$$
$$= \cos n\theta + i\sin n\theta$$

$w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ let $x^3 - 1 = 0$

$$w^3 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = \cos 2\pi + i\sin 2\pi = 1$$

$\therefore\ w^3 - 1 = 0$

$\therefore\ w$ satisfy the equation $x^3 - 1 = 0$

$$w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$w^r = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^r = \cos\frac{2r\pi}{3} + i\sin\frac{2r\pi}{3}$$
$$(w^r)^3 = \left(\cos\frac{2r\pi}{3} + i\sin\frac{2r\pi}{3}\right)^3 = \cos 2r\pi + i\sin 2r\pi = 1 \qquad r \in \mathbb{Z}$$
$$w^{2r} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{2r} = \cos\frac{4r\pi}{3} + i\sin\frac{4r\pi}{3}$$
$$(w^{2r})^3 = \left(\cos\frac{4r\pi}{3} + i\sin\frac{4r\pi}{3}\right)^3 = \cos 4r\pi + i\sin 4r\pi = 1$$

w^r and w^{2r} satisfy the equation $x^3 - 1 = 0$

$$1 + w^r + w^{2r} = 1 + \cos\frac{2r\pi}{3} + i\sin\frac{2r\pi}{3} + \cos\frac{4r\pi}{3} + i\sin\frac{4r\pi}{3}$$
$$= 1 + \cos\frac{2r\pi}{3} + \cos\frac{4r\pi}{3} + i\left(\sin\frac{2r\pi}{3} + \sin\frac{4r\pi}{3}\right)$$
$$= 1 + 2\cos r\pi\cos\frac{r\pi}{3} + 2i\left(\sin r\pi\cos\frac{r\pi}{3}\right)$$

$$= 1 + 2\cos r\pi \cos\frac{r\pi}{3} \qquad \text{since} \quad \sin r\pi = 0$$

When r even $\quad \cos r\pi = 1 \quad \cos\frac{r\pi}{3} = \frac{1}{2}$

When r odd $\quad \cos r\pi = -1 \quad \cos\frac{r\pi}{3} = \frac{1}{2} \qquad r$ is not a multiple of 3

$\therefore r \in \mathbb{Z} \qquad \cos r\pi \cos\frac{r\pi}{3} = -\frac{1}{2}$

$\therefore 1 + w^r + w^{2r} = 0$

$f(x) = 1 + x + x^2 + x^3$

$f(1) = 4$

$f(w) = 1 + w + w^2 + w^3$

Since $1 + w^r + w^{2r} = 0 \quad r \in \mathbb{Z} \qquad r$ is not a multiple of 3

$f(w) = 1$

$f(w^2) = 1 + w^2 + (w^2)^2 + (w^2)^3$

$$\begin{aligned} f(w) &= 1 + w^2 + w^4 + w^6 && w^3 = 1 \\ &= 1 + w^2 + w + w^3 && w^6 = 1 \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

$$(1 + x + x^2 + x^3)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + .. + a_{3n-2}x^{3n-2} + a_{3n-1}x^{3n-1} + a_{3n}x^{3n}$$

When $x = 1$

$4^n = a_0 + a_1 + a_2 + a_3 + a_4 + .. + a_{3n-2} + a_{3n-1} + a_{3n} --- (1)$

When $x = w$

$(1 + w + w^2 + w^3)^n = (0 + w^3)^n = w^{3n} = 1$

$1 = a_0 + a_1w + a_2w^2 + a_3w^3 + a_4w^4 + .. + a_{3n-2}w^{3n-2} + a_{3n-1}w^{3n-1} + a_{3n}w^{3n}$

$1 = a_0 + a_1w + a_2w^2 + a_3 + a_4w + .. + a_{3n-2}w + a_{3n-1}w^2 + a_{3n} \quad --- (2)$

When $x = w^2$

$(1 + w^2 + (w^2)^2 + (w^2)^3)^n = (1 + w^2 + w + w^3)^n = 1$

$$1 = a_0 + a_1w^2 + a_2w^4 + a_3w^6 + a_4w^8 + .. + a_{3n-2}w^{6n-4} + a_{3n-1}w^{6n-2} + a_{3n}w^{6n}$$

$1 = a_0 + a_1w^2 + a_2w^4 + a_3 + a_4w^2 + .. + a_{3n-2}w^2 + a_{3n-1}w + a_{3n} --- (3)$

$(1) + (2) + (3)$

$$4^n + 2 = 3a_0 + a_1(1 + w + w^2) + a_2(1 + w^2 + w) + 3a_3 + a_4(1 + w + w^2) + .. + a_{3n-2}(1 + w + w^2) + a_{3n-1}(1 + w^2 + w) + 3a_{3n}$$

Since $1 + w + w^2 = 0$

$3(a_0 + a_3 + a_6 + .. + a_{3n}) = 4^n + 2$

$a_0 + a_3 + a_6 + .. + a_{3n} = \frac{1}{3}(4^n + 2)$

$(1)+(2)\times w^2 + (3)\times w^3$

$4^n + w^2 + w = a_0(1+w^2+w) + a_1(1+w^3+w^3) + a_2(1+w+w^2)$

$+a_3(1+w+w^2) + a_4(1+w^3+w^3) + ..$

$..+a_{3n-2}\,(1+w^3+w^3) + a_{3n-1}\,(1+w+w^2) + a_{3n}(1+w^2+w)$

Since $1+w^3+w^3=0$ and $1+w^2+w=0$

$= (a_1+a_4+..+a_{3n-1})3$

$\frac{1}{3}(4^n-1) = (a_1+a_4+..+a_{3n-1})$

$(1)+(2)\times w + (3)\times w^2$

$4^n + w + w^2 = a_0(1+w+w^2) + a_1(1+w^2+w) + a_2(1+w^3+w^3)$

$+a_3(1+w+w^2) + a_4(1+w^2+w) + ..$

$..+a_{3n-2}\,(1+w^2+w) + a_{3n-1}\,(1+w^3+w^3) + 3a_{3n}(1+w+w^2)$

Since $1+w+w^2=0$ and $w^3=1$

$4^n + w + w^2 = (a_2+a_5+a_8..+a_{3n-2})$

$\frac{1}{3}(4^n-1) = (a_2+a_5+a_8+..+a_{3n-1})$

b) Let $z = x+iy$

$\arg(z+i) - \arg(z-i) = \arg\left(\frac{z+i}{z-i}\right)$

$\frac{z+i}{z-i} = \left\{\frac{x+(y+1)i}{x+(y-1)i}\right\} \quad \frac{x-(y+1)i}{x-(y-1)i} = \frac{x^2+(y^2-1)+x[(y+1)-(y-1)]i}{x^2+(y-1)^2}$

$= \frac{x^2+y^2-1}{x^2+(y-1)^2} + \frac{2xi}{x^2+(y-1)^2}$

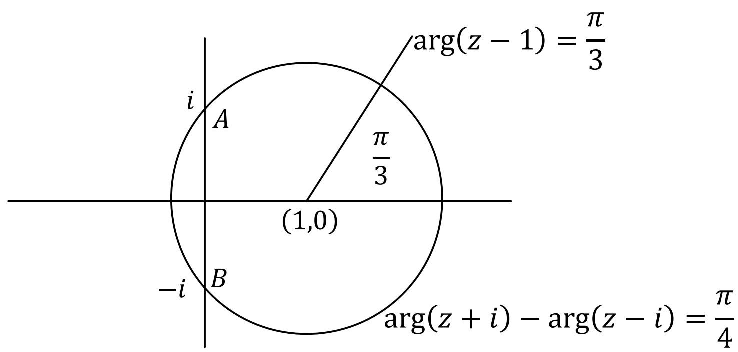
$\therefore \arg\left(\frac{z+i}{z-i}\right) = \theta \qquad \tan\theta = \frac{2x}{x^2+y^2-1}$

If

$\theta = \frac{\pi}{4} \qquad \tan\frac{\pi}{4} = 1$

$\therefore \frac{2x}{x^2+y^2-1} = 1$

$x^2+y^2-2x-1=0 \qquad c \equiv (1,0) \qquad r \equiv \sqrt{2}$



If $z=-i$ $\quad \frac{z+i}{z-i}=0 \quad \arg\left(\frac{z+i}{z-i}\right)=\arg(0)$ is not defined

When $z=i$ $\quad \frac{z+i}{z-i}$ is not defined.

$\therefore$ The locus of P is arc of a circle without the point A and B.

The centre of the circle $C\equiv(1,0)$ and the radius is $\sqrt{2}$

$\arg(z-1)=\frac{\pi}{3}$

$\therefore$ The locus of P is a straight line.

The equation $\frac{y}{x-1}=\sqrt{3}$

$y=\sqrt{3}(x-1),\ \ x>1$

$x^2+y^2-2x-1=0$

$(x-1)^2+y^2=2$

$\left(\frac{y}{\sqrt{3}}\right)^2+y^2=2$

$y^2+3y^2=6$

$4y^2=6$

$y=\pm\sqrt{\frac{3}{2}}$ $\qquad$ but $x>1$ $\quad \therefore\ y=\sqrt{\frac{3}{2}}$

$y=\sqrt{3}(x-1)$

$\sqrt{\frac{3}{2}}=\sqrt{3}(x-1)$

$x=1+\frac{1}{2}=\frac{3}{2}$

$\therefore\ \frac{3}{2}+\sqrt{\frac{3}{2}}\,i$ is the required complex number.

2.
a).

$$z = r\{\cos\theta + i\sin\theta\}$$
$$= r\{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\} \text{ where } k \in \mathbb{Z}$$
$$z_k = z^{\frac{1}{4}} = [r\{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}]^{\frac{1}{4}}$$
$$= r^{\frac{1}{4}}\left\{\cos\left(\frac{2k\pi + \theta}{4}\right) + i\sin\left(\frac{2k\pi + \theta}{4}\right)\right\}$$
$$\therefore {z_k}^4 = \left[r^{\frac{1}{4}}\left\{\cos\left(\frac{2k\pi + \theta}{4}\right) + i\sin\left(\frac{2k\pi + \theta}{4}\right)\right\}\right]^4$$
$$= r\left\{\cos 4\left(\frac{2k\pi + \theta}{4}\right) + i\sin 4\left(\frac{2k\pi + \theta}{4}\right)\right\}$$
$$= r\{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}$$
$$= r\{\cos\theta + i\sin\theta\}$$
$$= z \qquad \text{for } k = 0,1,2,3$$

$$z_0 = r^{\frac{1}{4}}\left\{\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right\}$$
$$z_1 = r^{\frac{1}{4}}\left\{\cos\left(\frac{2\pi + \theta}{4}\right) + i\sin\left(\frac{2\pi + \theta}{4}\right)\right\}$$
$$z_2 = r^{\frac{1}{4}}\left\{\cos\left(\frac{4\pi + \theta}{4}\right) + i\sin\left(\frac{4\pi + \theta}{4}\right)\right\}$$
$$z_3 = r^{\frac{1}{4}}\left\{\cos\left(\frac{6\pi + \theta}{4}\right) + i\sin\left(\frac{6\pi + \theta}{4}\right)\right\}$$

$$z_k = 16^{\frac{1}{4}} = 24^{\frac{1}{4}}\left\{\cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}\right\}$$
$$z_0 = 2\{\cos 0 + i\sin 0\} = 2$$
$$z_1 = 2\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\} = 2i$$
$$z_2 = 2\{\cos\pi + i\sin\pi\} = -2$$
$$z_3 = 2\left\{\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right\} = -2i$$
$$\left(16^{\frac{1}{4}}\right)^2 = 2^2 = 4$$
$$= (2i)^2 = -4$$
$$= (-2)^2 = 4$$
$$= (-2i)^2 = -4$$

$$\left(16^{\frac{1}{4}}\right)^2 = 256 = 4^4$$

$$16 = 2^4(\cos 2k\pi + i\sin 2k\pi)$$
$$16^2 = 4^4(\cos 4k\pi + i\sin 4k\pi)$$

$$(16^2)^{\frac{1}{4}} = [4^4(\cos 4k\pi + i\sin 4k\pi)]^{\frac{1}{4}}$$
$$= 4(\cos k\pi + i\sin k\pi)$$

$z_0 = 4$
$z_1 = -4$
$z_2 = 4$
$z_3 = -4$

$\therefore$ The set of values of $\left(16^{\frac{1}{4}}\right)^2$ is a proper subset of values of $(16^2)^{\frac{1}{4}}$

b)

$$z = \frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + i\left(\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - i\left(\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\right)}$$
$$= \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2 + i\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2 - i\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}$$
$$= \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) + i\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) - i\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}$$
$$= \frac{\left(\cos\frac{\pi}{4}\cos\frac{\theta}{2} + \sin\frac{\pi}{4}\sin\frac{\theta}{2}\right) + i\left(\cos\frac{\pi}{4}\sin\frac{\theta}{2} - \cos\frac{\pi}{4}\sin\frac{\theta}{2}\right)}{\left(\cos\frac{\pi}{4}\cos\frac{\theta}{2} + \sin\frac{\pi}{4}\sin\frac{\theta}{2}\right) - i\left(\sin\frac{\pi}{4}\cos\frac{\theta}{2} - \cos\frac{\pi}{4}\sin\frac{\theta}{2}\right)}$$
$$= \frac{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}$$
$$= \left[\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]\left[\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$
$$= \cos\left(\frac{\pi}{4} - \frac{\theta}{2} + \frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2} + \frac{\pi}{4} - \frac{\theta}{2}\right)$$
$$= \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$$
$$= \sin\theta + i\cos\theta$$

i) $\left(1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5 = 0$

$$= \left[\left(\frac{1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}}{1 + \sin\frac{\pi}{5} + i\sin\frac{\pi}{5}}\right)^5 + i\right] + i\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5$$
$$= \left[\left(\sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\right]\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5$$
$$= \left[\left(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}\right)^5\right] + i\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5$$

$$= \left[\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) + i\right]\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5$$
$$= [(0 - i) + i]\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5$$
$$= 0$$

ii.

$$z = \left\{\frac{\sqrt{2} + 1 + i}{\sqrt{2} + 1 - i}\right\}^{\frac{1}{4}} = \left\{\frac{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}\right\}^{\frac{1}{4}} = \left\{\frac{1 + \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}}{1 + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}}\right\}^{\frac{1}{4}}$$
$$= \left(\sin\frac{\pi}{4} + i\cos\frac{\pi}{4}\right)^{\frac{1}{4}} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{\frac{1}{4}}$$
$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{\frac{1}{4}}$$
$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right)\frac{1}{4} + i\sin\left(2k\pi + \frac{\pi}{4}\right)\frac{1}{4}\right]$$

$$z_0 = \cos\frac{\pi}{16} + i\sin\frac{\pi}{16}$$

$$z_1 = \cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}$$

$$z_2 = \cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}$$

$$z_3 = \cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}$$

3.

$$\mathbb{R}e\left(\frac{1}{z+1}\right) = \frac{1}{2}$$

Let $z = x + iy$

$$z + 1 = (x + 1) + iy$$

$$\frac{1}{z+1} = \frac{1}{(x+1) + iy} \times \frac{(x+1) - iy}{(x+1) - iy}$$
$$\frac{1}{z+1} = \frac{(x+1) - iy}{(x+1)^2 + y^2} == \frac{(x+1)}{(x+1)^2 + y^2} - i\frac{y}{(x+1)^2 + y^2}$$
$$\therefore \mathbb{R}e\left(\frac{1}{z+1}\right) = \frac{(x+1)}{(x+1)^2 + y^2} = \frac{1}{2}$$
$$2x + 2 = (x+1)^2 + y^2$$
$$x^2 + y^2 = 1$$

When $z = -1 \Leftrightarrow x = -1, y = 0$ satisfy the equation $x^2 + y^2 - 1 = 0$

But when $z = -1$, $\frac{1}{z+1}$ is not defined

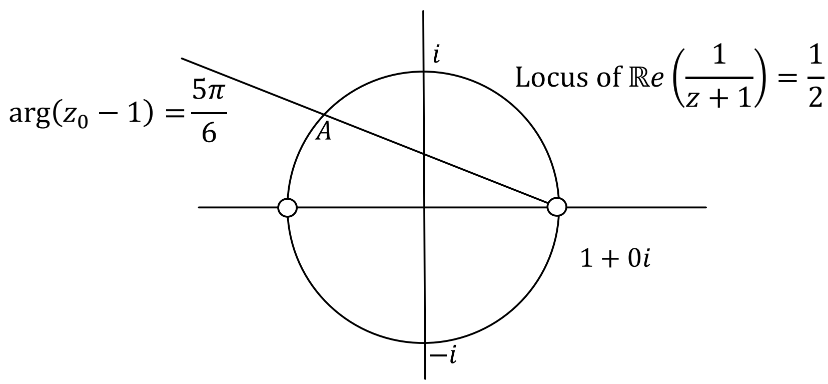
$\therefore \mathbb{R}e\left(\frac{1}{z+1}\right) = \frac{1}{2}$

z representing the circle $x^2 + y^2 = 1$ without the point $(-1,0)$

$$\arg(z_0 - 1) = \frac{5\pi}{6} \Leftrightarrow \frac{y_0}{x_0 - 1} = \tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$$

$$y_0 = -\frac{1}{\sqrt{3}}(x_0 - 1); \quad x_0 < 1$$

Locus of $\mathbb{R}e\left(\frac{1}{z+1}\right) = \frac{1}{2}$



$$y_0 = -\frac{1}{\sqrt{3}}(x_0 - 1)$$
$${x_0}^2 + {y_0}^2 = 1$$
$${x_0}^2 + \left\{-\frac{1}{\sqrt{3}}(x_0 - 1)\right\}^2 = 1$$
$${x_0}^2 + \frac{1}{3}(x_0 - 1)^2 = 1$$
$$3{x_0}^2 + (x_0 - 1)^2 = 3$$
$$2{x_0}^2 - x_0 - 1 = 0$$
$$(2x_0 + 1)(x_0 - 1) = 0$$

$$x_0 \neq 1 \qquad x_0 = -\frac{1}{2} \qquad y_0 = -\frac{1}{\sqrt{3}}\left(-\frac{1}{2} - 1\right) = \frac{\sqrt{3}}{2}$$

$$z_A = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}i\right)$$

b) $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$|z_1| = r_1 \qquad |z_2| = r_2$

$z_1 - z_2 = (r_1 \cos\theta_1 - r_2\cos\theta_1) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$|z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2\cos\theta_1)^2 + i(r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$|z_1 - z_2|^2 = {r_1}^2 + {r_2}^2 - 2r_1 r_2\cos(\theta_1 - \theta_2) - - - (A)$

$|1 - \overline{z_2}z_1|^2, \qquad \overline{z_2} = r_2(\cos\theta_2 + i\sin -\theta_2)$

$z_1\,\overline{z_2} = r_1 r_2[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

For $|1 - \overline{z_2}z_1|^2$ by comparing with (A) $r_1 = 1 \qquad \theta_1 = 0$

$r_1 = r_1 r_2 \qquad \theta_2 = \theta_1 - \theta_2$

$|1 - \overline{z_2}z_1|^2 = 1 + (r_1 r_2)^2 - 2 \times 1 \times r_1 r_2\big[\cos\big(0 - (\theta_1 - \theta_2)\big)\big]$

$|1 - \overline{z_2}z_1|^2 = 1 + {r_1}^2{r_2}^2 - 2r_1 r_2[\cos(\theta_1 - \theta_2)] - - - (B)$

$(B) - (A)$

$$\begin{aligned}|1 - \overline{z_2}z_1|^2 - |z_1 - z_2|^2 &= 1 + {r_1}^2{r_2}^2 - 2r_1 r_2[\cos(\theta_1 - \theta_2)] \\ &\qquad - [{r_1}^2 + {r_2}^2 + r_1 r_2\cos(\theta_1 - \theta_2)] \\ &= 1 + {r_1}^2{r_2}^2 - {r_1}^2 - {r_2}^2 \\ &= (1 - {r_1}^2)(1 - {r_2}^2)\end{aligned}$$

$|1 - \overline{z_2}z_1|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$

c)

i) $|z + 3i|^2 - |z - 3i|^2 = 10$

$z = x + iy$

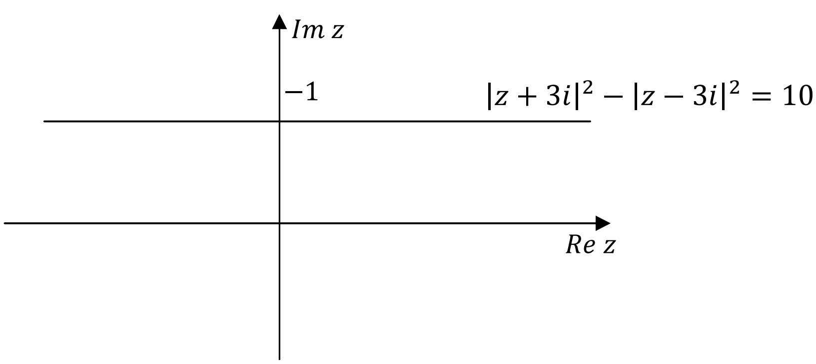
$|z + 3i|^2 = x^2 + (y + 3)^2$

$|z - 3i|^2 = x^2 + (y - 3)^2$

$|z + 3i|^2 - |z - 3i|^2 = x^2 + (y + 3)^2 - x^2 + (y - 3)^2 = 10$

$12y = 10$

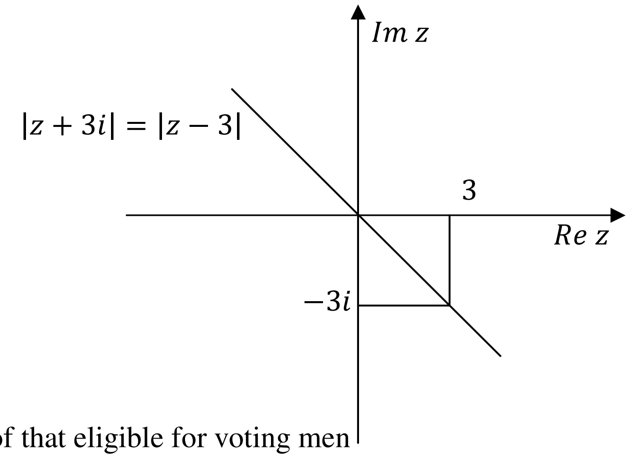
$y = \dfrac{5}{6}$



ii. $|z + 3i| = |z - 3|$

$x^2 + (y + 3)^2 = (x - 3)^2 + y^2$

$x + y = 3$



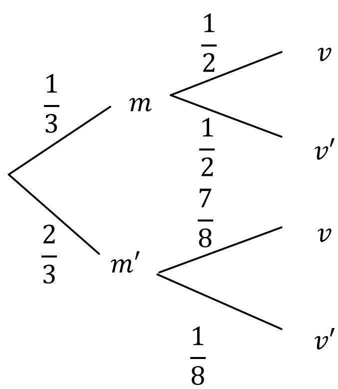
4.

a) $P(m) =$ the probability of that eligible for voting men

$P(m') =$ the probability of that eligible for voting women

$P(m) + P(m') = 1$ $\qquad$ $P(m') = 2\,P(m)$

$\therefore\ P(m) = \frac{1}{3} \qquad P(m') = \frac{2}{3}$



$P(v/m) = \dfrac{1}{2}$

$P(v/m') = \dfrac{7}{8}$

$$P(v) = P(v/m)p(m) + P(v/m')p(m')$$
$$= \frac{1}{2} \times \frac{1}{3} + \frac{2}{3} \times \frac{7}{8}$$
$$= \frac{18}{24}$$
$$= \frac{3}{4}$$

i. For the random group of 4 inhabitants the probability that just one of the votes

$$= \quad {}^4C_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^3$$

$$= 4 \times \frac{3}{256} = \frac{3}{64}$$

ii. The probability that two or more vote $= 1 - \left[\ {}^4C_0\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 + {}^4C_1\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1\right]$

$$= 1 - \left[4 \times \frac{1}{256} + 4 \times \frac{3}{256}\right]$$

$$= 1 - \frac{13}{256}$$

$$= \frac{243}{256}$$

iii. Probability that man votes for married couple $P(A) = \frac{4}{5}$

Probability that woman votes for married couple $P(B) = \frac{6}{7}$

Given that $P(A/B) = \frac{21}{25} \quad P(B/A) = \frac{9}{10}$

$P(A) = P(A \cap B\ \cup\ A \cap B') = P(A \cap B) + P\ (A \cap B')$

$\frac{4}{5} = P(A \cap B) + P\ (A \cap B') - - - (1)$

$P(B) = P(A \cap B\ \cup\ A' \cap B) = P(A \cap B) + P\ (A' \cap B)$

$\frac{9}{10} = P(A \cap B) + P\ (A' \cap B) - - - (2)$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{21}{25} \times \frac{6}{7} = P(A \cap B)$$

$$\therefore P(A \cap B) = \frac{18}{25}$$

The probability that husband and wife both vote $= \frac{18}{25}$

iv. From (1)

$$P\ (A \cap B') = \frac{4}{5} - P(A \cap B)$$

$$= \frac{4}{5} - \frac{18}{25} = \frac{2}{25}$$

$\therefore$ The probability that a husband vote and his wife does not vote $= \frac{2}{25}$

The probability that a wife vote and his husband does not vote $P(A' \cap B)$

$$= \frac{6}{7} - \frac{18}{25} = \frac{24}{175}$$

No of vote x	0	1	2
Probability $P(x)$	$\frac{11}{175}$	$\frac{2}{25}+\frac{24}{175}$ $=\frac{38}{175}$	$\frac{126}{175}$

$$\text{Expectation} = \sum_{i=0}^{2}\{p(x_i)\}(x_i)$$
$$= 0 \times \frac{11}{175} + \frac{38}{175} \times 1 + \frac{126}{175} \times 2$$
$$= \frac{290}{175}$$
$$= \frac{58}{35}$$

b. 3 3 3 4 4 4 4 5 5

x	6	7	8	9	10
$P(x)$	$\frac{3}{36}$	$\frac{12}{36}$	$\frac{12}{36}$	$\frac{8}{36}$	$\frac{1}{36}$

$$P(6) = \frac{{}^3C_2}{{}^9C_2} = \frac{3}{36} = \frac{1}{12}$$

$$P(7) = \frac{{}^3C_1 \times {}^4C_1}{{}^9C_2} = \frac{12}{36} = \frac{1}{3}$$

$$P(8) = \frac{{}^3C_1 \times {}^2C_1 \times {}^4C_2}{{}^9C_2} = \frac{12}{36}$$

$$P(9) = \frac{{}^2C_1 \times {}^4C_1}{{}^9C_2} = \frac{8}{36} = \frac{2}{9}$$

$$P(10) = \frac{{}^2C_2}{{}^9C_2} = \frac{1}{36}$$

$$E(x) = \mu = \frac{6 \times 3 + 7 \times 12 + 8 \times 12 + 9 \times 8 + 10 \times 1}{36}$$

$$E(x) = \frac{280}{36} = \frac{70}{9}$$

$$Var(x) = \sigma^2 = \sum(x_r - \mu)P(x_r)$$

$$Var(x) = \sigma^2 = E(x^2) - [E(x)]^2$$

$$= \frac{36 \times 3 + 49 \times 12 + 64 \times 12 + 81 \times 8 + 100 \times 1}{36} - \left(\frac{70}{9}\right)^2$$

$$= 0.951$$

$$\sigma = \sqrt{0.951}$$

$$= 0.975$$

The probability that the second value of x greater than or equal to the first value of x

$$= \frac{3}{36} \times \frac{3}{36} + \frac{12}{36} \times \frac{15}{36} + \frac{12}{36} \times \frac{21}{36} + \frac{8}{36} \times \frac{35}{36} + \frac{1}{36} \times \frac{36}{36}$$

$$= \frac{9 + 180 + 324 + 280 + 36}{36 \times 36}$$

$$= \frac{829}{36 \times 36} = 0.640$$

c.

i. $F(X > x) = ax^2 + b \qquad 0 \leq x \leq 4$

$F(X > 0) = 1 \qquad F(X > 4) = 0$

$a0^2 + b = 1 \qquad a4^2 + b = 1$

$b = 1 \qquad a = \frac{-1}{16}$

$$F(x) = \left(1 - \frac{1}{16}x^2\right)$$

$$F(x) = \begin{cases} \frac{x^2}{16} & 0 \leq x \leq 4 \\ 1 & x > 4 \end{cases}$$

ii. The probability density function $= f\ (x) = \frac{dF(x)}{dx}$

$$= \frac{2x}{16}$$

$$= \frac{x}{8}$$

$$f(x) = \begin{cases} \frac{x}{8} & 0 \leq x \leq 4 \\ 0 & \text{othewise} \end{cases}$$

iii.

$$E(x) = \int xf(x)dx = \int_0^4 x\frac{x}{8}dx = \left[\frac{x^3}{24}\right]_0^4 = \frac{64}{24} = \frac{8}{3}$$

iv.

$$\sigma^2 = \int_0^4 \left(x - \frac{8}{3}\right)^2 f(x)dx = \int_0^4 \left(x - \frac{8}{3}\right)^2 \frac{x}{8} dx$$

$$= \int_0^4 \left(\frac{x^3}{8} - \frac{2x^2}{3} + \frac{8x}{9}\right)^2 dx = \left[\frac{x^4}{32} - \frac{2x^3}{9} + \frac{8x^2}{18}\right]_0^4$$

$$= \frac{256}{32} - \frac{128}{9} + \frac{128}{18}$$

$$= 8 - \frac{128}{9} + \frac{64}{9} = \frac{8}{9}$$

$$\sigma_x = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

v.

$$P_{\mu-\sigma\leq x\leq\mu+\sigma} = P_{\frac{8-2\sqrt{2}}{3}\leq x\leq\frac{8+2\sqrt{2}}{3}} = \int_{\frac{8-2\sqrt{2}}{3}}^{\frac{8+2\sqrt{2}}{3}} \frac{x}{8} dx$$

$$\left[\frac{x^2}{16}\right]_{\frac{8-2\sqrt{2}}{3}}^{\frac{8+2\sqrt{2}}{3}} = \frac{1}{16}\left[\left(\frac{8+2\sqrt{2}}{3}\right)^2 - \left(\frac{8-2\sqrt{2}}{3}\right)^2\right]$$

$$= \frac{1}{144}(64)\sqrt{3} = \frac{4\sqrt{3}}{9}$$